# PHY 604: Computational Methods in Physics and Astrophysics II

Homework #5

Due: Nov. 28, 2023

*Programs can be written in any language, In addition to the program, you should have a writeup that contains the plots requested in the homework questions, answers to any analytical or explanation questions, and a short description of your code and how to run it. This can be done via, e.g., LaTeX, markdown, jupyter notebooks, etc.*

*Code and writeup should be submitted using* `git` *via github in the repo that was created from the* github classroom *link.*

1. *Relaxation methods for the Poisson equation* (based on Garcia) Write a program that solves the two-dimensional Poisson equation in a square geometry with the Dirichlet boundary conditions $\Phi = 0$ at the boundaries using one of the relaxation methods discussed in class (i.e., Jacobi, Gauss-Seidel, or SOR). Map the potential for a single charge at the center of the system. Compare with the potential for a charge in free space (recall that a point charge in 2D is equivalent to a line charge in 3D).

2. *Stability analysis* (based on Garcia) In class we discussed two methods for assessing the stability of PDEs, *von Neumann stability analysis* and *matrix stability analysis*. In this problem, we will explore these approaches on a variety of explicit/implicit schemes. Note: Several parts of this problem do not require writing any programs.

   **(a)** Consider the leapfrog scheme for solving the advection equation introduced in question 3(b) of Homework 4:
   $$\frac{a_i^{n+1} - a_i^{n-1}}{2\tau} = -c\frac{a_{i+1}^n - a_{i-1}^n}{2h}. \tag{1}$$
   Use the von Neumann stability analysis to show that the method is stable only if $\tau \leq h/|c|$.

   **(b)** The Lax scheme for the advection equation with periodic boundary conditions may be written as
   $$\mathbf{a}^{n+1} = \left(\frac{1}{2}\mathbf{C} - \frac{c\tau}{2h}\mathbf{B}\right)\mathbf{a}^n \equiv \mathbf{A}\mathbf{a}^n \tag{2}$$
   where
   $$\mathbf{a}^n = \begin{bmatrix} a_0^n \\ a_1^n \\ a_2^n \\ \vdots \\ a_{N-1}^n \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & -1 \\ -1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & \text{-1} & 0 \end{bmatrix}. \tag{3}$$
   Demonstrate that the matrix stability analysis gives by $\tau \leq h/|c|$ by plotting the spectral radius versus time step of $\mathbf{A}$.

3. *Radioactive decay chain* (Based on Newman exercise 10.2) The isotope $^{213}$Bi decays to a much more stable isotope $^{209}$Bi via one of two different routes, with probabilities and half-lives shown in Fig. 1. Starting with a sample consisting of 10,000 atoms of $^{213}$Bi, simulate the decay of atoms over 20,000 s using a time slices of 1 s. **Hint:** Start from the lower decay processes ($^{209}\text{Pb} \rightarrow^{209}$ Bi) and work your way up.

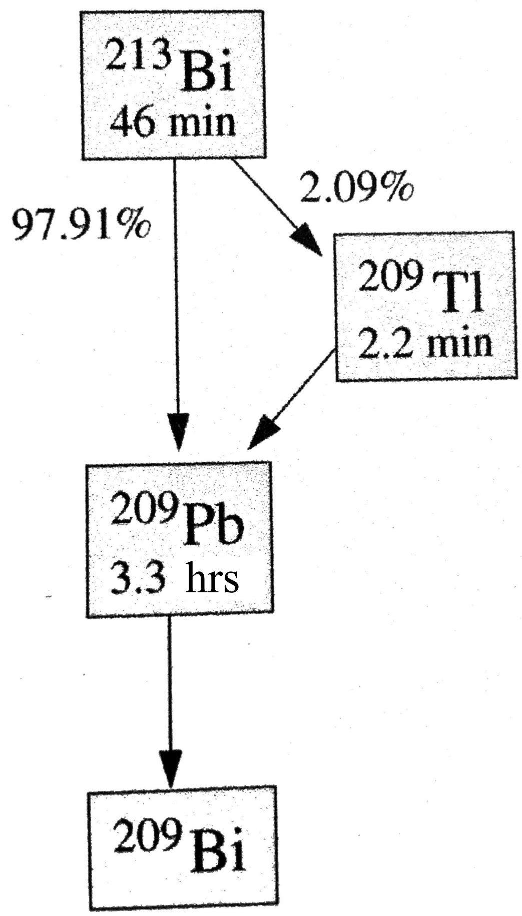


Figure 1: Decay paths from $^{213}$Bi to $^{209}$Bi.